Definability of henselian valuations in positive (residue) characteristic

Margarete Ketelsen and Simone Ramello
joint with Piotr Szewczyk (Dresden)

Institut für Mathematische Logik und Grundlagenforschung
University of Münster, Germany

PhDs in Logic, Granada, 04.10.2023

MM
Mathematics
Münster
Cluster of Excellence

The road ahead

$$\underbrace{\text{model theory}}_{\text{mathematical logic}} \text{ of } \underbrace{\text{valued fields}}_{\text{algebra}}.$$

Today, we will try to:

- Tell you what valued fields are.
- Give you an idea of what our results look like.
- Tell you about an obstacle in this area and how we turned it into a tool.

Definition

A valuation on a field K is a surjective map $v\colon K^\times \to \Gamma$, where $(\Gamma, +, \leqslant, 0)$ is an ordered abelian group, such that:

- $v(xy) = v(x) + v(y)$, *multiplying two elements sums their valuations*
- $v(x+y) \geqslant \min\{v(x), v(y)\}$. *all triangles are isosceles*

(Counter)intuition: an element $r \in K^\times$ is *large* if if its valuation $v(r) \in \Gamma$ is *small*, i.e. close to 0. Along this intuition, we usually set $v(0) \coloneqq \infty$.

The ordered abelian group Γ is called the *value group*. We also denote it by vK.

Our favourite example

Fix a prime number p.

- If $a \in \mathbb{Z} \setminus \{0\}$, then
$$v_p(a) \coloneqq \max\{n \in \mathbb{N} \colon p^n \text{ divides } a\}.$$
For example, $v_3(6560) = 0$. According to v_3, then, 6560 is “big”. But $v_3(6561) = 8$, which is then “smaller” than 6560. If $a, b \in \mathbb{Z} \setminus \{0\}$ are coprime, then
$$v_p\left(\frac{a}{b}\right) \coloneqq v_p(a) - v_p(b).$$
- This defines a valuation $v_p \colon \mathbb{Q} \setminus \{0\} \to \mathbb{Z}$, called *the p-adic valuation*. With it, we can define a distance on $\mathbb{Q}$ by setting $d_p(a, b) \coloneqq p^{-v_p(a-b)}$.
- If we *complete* the corresponding metric space, we obtain a (new) valued field called $\mathbb{Q}_p$, with its own valuation v_p. These are the *p-adic numbers*.

Why you should like the p-adics

- $(\mathbb{Q}_p, v_p)$ is crucial for algebraic purposes. But we are logicians (allegedly)!
- A valuation is "the same" as its valuation ring, i.e. the subring

$$\mathcal{O}_v = \{x \in K \mid v(x) \geqslant 0\}.$$

  This is the part where we should tell you that ∞ is larger than all elements of Γ, and thus $0 \in \mathcal{O}_v$.
- In the case of $\mathbb{Q}_p$, this subring is called $\mathbb{Z}_p$ (guess why!). Julia Robinson pointed out something remarkable about $\mathbb{Z}_p$ (for $p \neq 2$):

$$\mathbb{Z}_p = \{x \in \mathbb{Q}_p \mid \exists Y(Y^2 = 1 + px^2)\}.$$

  There is a similar formula for $p = 2$.
- $\mathbb{Z}_p$ is given, as a subset of $\mathbb{Q}_p$, by a polynomial equation together with some quantifiers. We say that it is a *definable* set in the language of rings.

Logicians, assemble! cont'd

Big question: Is this common? When is some valuation ring definable in the language of rings?

The problem of henselianity

Not all valuations are created equal.

- Take a field K with a valuation v. We give you an algebraic extension L of K, e.g. $L = K(\alpha)$ where α is the root of some polynomial over K. Can you extend v to L?
  Yes, but often in several different ways.
- v is *henselian* if there is a **unique** way to extend v to any algebraic extension of K. A henselian valuation is a bit like a *fill the gaps* exercise in a textbook.
- v_p is henselian. *We will only care about henselian valuations.*

The big question, take 2

Big question: when is
a henselian valuation ring definable in the language of rings?

Two fields in disguise

- To any valued field (K, v) we can associate another "smaller" field, called the *residue field*,

$$Kv \coloneqq {}^{\{x \in K \colon v(x) \geqslant 0\}} \big/ {}_{\{x \in K \colon v(x) > 0\}}.$$

Indeed, $\mathfrak{m}_v \coloneqq \{x \in K \colon v(x) > 0\}$ is the unique maximal ideal of $\mathcal{O}_v = \{x \in K \mid v(x) \geqslant 0\}$.

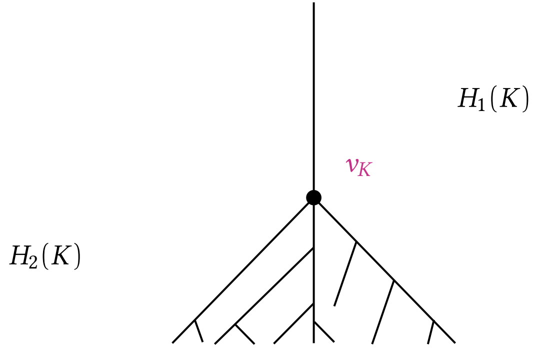
- Example: $(\mathbb{Q}, v_p)$ and $(\mathbb{Q}_p, v_p)$ both have residue field $\mathbb{F}_p$, the finite field with p elements
In fact, $\mathbb{Q} \subseteq \mathbb{Q}_p$ is an *immediate extension*: They have the same value groups and residue fields.
- So a valued field consists of *two fields*: the "big" valued field and the "smaller" residue field. If we talk about the characteristic of a valued field, we talk about the characteristics of the two fields
  - equicharacteristic zero: $\mathrm{char}(K) = \mathrm{char}(Kv) = 0$
  - mixed characteristic: $\mathrm{char}(K) = 0 < p = \mathrm{char}(Kv)$, where p is prime
  - positive characteristic: $\mathrm{char}(K) = \mathrm{char}(Kv) = p$, where p is prime

A canonical friend

- Henselian valuations on a given field K arrange themselves nicely according to whether their residue field is separably closed or not,

$$H_1(K) \coloneqq \{v\colon Kv \text{ is not separably closed}\} \text{ vs. } H_2(K) \coloneqq \{v\colon Kv \text{ is separably closed}\}.$$

- $H_1(K)$ is linearly ordered by inclusion.
  The "middle point" between $H_1(K)$ and $H_2(K)$ is the *canonical henselian valuation* v_K.



The gist of it

$$\underbrace{\exists \text{ definable (non-trivial) henselian valuation}}_{\text{Logic question}} \iff \underbrace{\text{Conditions on the canonical henselian valuation}}_{\text{(Almost) algebra answer}}$$

What we proved

Theorem (Jahnke, Koenigsmann, 2017; Ketelsen, Ramello, Szewczyk, 2023)

Let K be a non-separably closed henselian field.
If $\mathrm{char}(K) = p > 0$, *then assume that* K *is perfect.*
If $\mathrm{char}(K) = 0 < p = \mathrm{char}(Kv_K)$, *then assume that* $\mathcal{O}_{v_K}/p$ *is semi-perfect.*
Then,

$$K \text{ admits a definable non-trivial henselian valuation} \iff \begin{cases} Kv_K = Kv_K^{\mathrm{sep}}, & \text{or} \\ Kv_K \text{ is not t-henselian}, & \text{or} \\ \exists L \succeq Kv_K \text{ with } v_L L \text{ divisible}, & \text{or} \\ v_K K \text{ is not divisible}, & \text{or} \\ (K, v_K) \text{ is not defectless}, & \text{or} \\ \exists L \succeq Kv_K \text{ with } (L, v_L) \text{ not defectless}. & \end{cases}$$

What we had before

Theorem (Jahnke, Koenigsmann, 2017~~; Ketelsen, Ramello, Szewczyk, 2023~~)

Let K be a non-separably closed henselian field, $\mathrm{char}(Kv) = 0$.
~~*If* $\mathrm{char}(K) = p > 0$, *then assume that K is perfect.*~~
~~*If, further,* $\mathrm{char}(K) = 0 < p = \mathrm{char}(Kv_K)$, *then further assume that* $\mathcal{O}_{v_K}/p$ *is semi-perfect.*~~
Then,

$$K \text{ admits a definable non-trivial henselian valuation} \iff \begin{cases} Kv_K = Kv_K^{\mathrm{sep}}, & \text{or} \\ Kv_K \text{ is not t-henselian}, & \text{or} \\ \exists L \succeq Kv_K \text{ with } v_L L \text{ divisible}, & \text{or} \\ v_K K \text{ is not divisible}, & \text{or} \\ \sout{(K, v_K) \text{ is not defectless},} & \text{or} \\ \sout{\exists L \succeq Kv_K \text{ with } (L, v_L) \text{ not defectless.}} \end{cases}$$

We don't talk about defect

Actually, we do now.

- Given a henselian valuation v and a finite field extension $K \subseteq L$, then there is a unique extension of v to L, which we denote by v again. Then, we have

$$[L : K] \geqslant [Lv : Kv](vL : vK).$$

  More precisely,

$$[L : K] = p^d [Lv : Kv](vL : vK),$$

  where $p = \mathrm{char}(Kv)$, if the latter is positive, and $p = 1$ if $\mathrm{char}(Kv) = 0$.

- We say that $(K, v) \subseteq (L, v)$ is *defectless* if

$$[L : K] = [Lv : Kv](vL : vK).$$

  In particular, then, if $\mathrm{char}(Kv) = 0$, then $p = 1$ and so equality holds. Otherwise, not being defectless (= *having defect*) is a problem.

- For us, however, defect is a **source of information**! (At least when it is "of independent type").